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Non-Hermitian harmonic oscillator with discrete complex or real spectrum for non-unitary squeeze operators

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Abstract

In the present paper, we enlarge the method of Debergh, Beckers and Szafraniec for non-Hermitian Hamiltonians with complex parameters and obtain real eigenvalues. For the case of non-unitary squeeze operators with two complex parameters of non-Hermitian harmonic oscillator, we obtain discrete complex spectrum and for special values of the complex parameters, the spectrum is discrete real and positive even though the corresponding operators are not *PT* symmetric.

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1. Introduction

Recently, Debergh *et al* [1] discussed the non-Hermitian position and momentum operators, leading to the generalized Heisenberg relations. The non-Hermitian oscillator with real spectra has been recently studied [2, 3] and exploited [4–6] in an intensive way, leading to new trends in fundamental quantum mechanics. According to [4–6], the authors replaced the well-known Hermiticity relation $\mathcal{H} = \mathcal{H}^{\dagger}$ by the weaker and more physical requirement $\mathcal{H}^{\ddagger} = \mathcal{H}$, where $\ddagger$ represents combined parity reflection and time reversal *PT*, and obtained new classes of complex Hamiltonians whose spectra are still real and positive. These *PT* symmetric theories may be viewed as analytic continuations of conventional theories from real to complex phase space. Also the authors of [1] in the papers [2, 3] studied the oscillator-like Hamiltonians of squeezing.

Quantum systems characterized by non-Hermitian Hamiltonian are of interest in several areas of theoretical physics [7]. Several authors have also studied [8, 9] the standard one-dimensional Schrödinger Hamiltonians with complex-valued potentials giving rise to a real energy spectrum. The simple example of the harmonic oscillator with complex mass leads to a non-Hermitian Hamiltonian with real spectrum, i.e. $\hbar\omega\big(n+\frac{1}{2}\big)$ and the uncertainty principle of Heisenberg holds. The case where the parameters of the generalized harmonic oscillator are time-dependent and the Hamiltonian is either Hermitian or non-Hermitian, has been studied by Baskoutas and co-workers [10–12] in a series of publications, mainly using the method of time evolution and the existence of squeezing.

Recently in a separate paper [13], the displaced squeezed number states of the phonon field in polar semiconductors has been examined. Also in another paper [14] entitled 'Non-unitary Weyl and squeeze operators', new states which can be considered as a generalization of the usual coherent and squeezed states have been constructed.

The present paper is organized as follows. In section 2, we apply the method of [1] with complex parameters and prove the existence of a real spectrum. In section 3, we study non-unitary squeezed states with two complex parameters with the help of the non-Hermitian harmonic oscillator and obtain a discrete complex spectrum or a real spectrum for specific cases, where the uncertainty principle holds. Finally, section 4 is devoted to concluding remarks.

2. Cases of non-Hermitian harmonic oscillator

Before studying the non-Hermitian harmonic oscillator, we extend the method of [1] for complex parameters, where the operators A and A^+ are not adjoint, i.e.

$$A=\frac{\mathrm{d}}{\mathrm{d}x}+f(x)\qquad A^+=x+b\qquad [A,A^+]=1\qquad (A^+)^\dagger\neq A \tag{1}$$

with the complex function $f(x)=f_1(x)+\mathrm{i}f_2(x)$ and b a real parameter. The complex function $f(x)$ must satisfy the existence of the integral $F(x)=\int f(x)\,\mathrm{d}x$ so that $\mathrm{e}^{-F(x)}\to 0$ for $x=\pm\infty$.

For $f(x)=cx^3$, $c>0$ and $b=0$, we get the example of [1], while for $c=c_1+\mathrm{i}c_2$, $c_1>0$ and $b\neq 0$, we obtain the eigenfunctions

$$\Psi_n(x)=\tilde{N}_n(x+b)^n\,\mathrm{e}^{-\frac{c}{4}x^4} \tag{2}$$

of the non-Hermitian operator

$$H=A^+A+\tfrac{1}{2} \tag{3}$$

which satisfy the equation

$$\left((x+b)\left(\frac{\mathrm{d}}{\mathrm{d}x}+cx^3\right)+\frac{1}{2}\right)\Psi_n(x)=\left(n+\frac{1}{2}\right)\Psi_n(x)\qquad n=0,1,2,\ldots \tag{4}$$

and the relations

$$A\Psi_0(x)=0\qquad \Psi_n(x)=\tilde{N}_n(x+b)^n\,\mathrm{e}^{-\frac{c}{4}x^4}=(\tilde{N}_n/\tilde{N}_0)(A^+)^n\Psi_0(x). \tag{5}$$

The normalization factor $\tilde{N}_n$ is given by the expression

$$\tilde{N}_n=\left\{\sum_{l=0}^{n}\frac{1}{(2l)!}\left(\frac{\mathrm{d}^{2l}}{\mathrm{d}b^{2l}}b^{2n}\right)\frac{1}{N_l^2}\right\}^{-\frac{1}{2}} \tag{6}$$

where

$$N_l=\frac{\sqrt{2}\left(\frac{c_1}{2}\right)^{\frac{2l+1}{8}}}{\sqrt{\Gamma\left(\frac{2l+1}{4}\right)}} \tag{7}$$

is the normalization factor for $b=0$.

Another simple example is as follows:

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + \lambda x \qquad A^{+} = x \qquad [A, A^{+}] = 1 \qquad \mathrm{Re}\,\lambda > 0 \tag{8}$$

with the eigenfunctions

$$\Psi_n(x) = \left(\frac{\lambda_1}{\pi}\right)^{\frac{1}{4}} \frac{\lambda_1^{\frac{1}{2}(n+\frac{1}{2})}}{\sqrt{\Gamma\left(n+\frac{1}{2}\right)}}\, \mathrm{e}^{-\frac{\lambda}{2}x^2} x^n \tag{9}$$

which satisfy the corresponding relations (5).

The mean values of the operators $\hat{x} = x$, $\hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and the dispersion relations are given by

$$\begin{aligned} &\bar{x} = 0 \qquad \bar{p} = 0 \qquad \bar{x}^2 = (\Delta x)^2 = \frac{n+1/2}{\lambda_1} \\ &\bar{p}^2 = (\Delta p)^2 = \lambda_1\left[\frac{n-1/4}{n-1/2} + \frac{\lambda_2^2}{\lambda_1^2}(n+1/2)\right]. \end{aligned} \tag{10}$$

From the above results, we have

$$(\Delta x)^2(\Delta p)^2 = \left[\frac{(n+1/2)(n-1/4)}{n-1/2} + \frac{\lambda_2^2}{\lambda_1^2}(n+1/2)^2\right] \geqslant \frac{1}{4}. \tag{11}$$

For $n = 0$, the above relation takes the form

$$(\Delta x)(\Delta p) = \frac{1}{2}\sqrt{1 + \frac{\lambda_2^2}{\lambda_1^2}} = \frac{1}{2}\frac{|\lambda|}{\lambda_1} \geqslant \frac{1}{2}. \tag{12}$$

In the following, we consider the adjoint operators

$$\begin{aligned} B &= \frac{1}{\sqrt{2\lambda_1}}(\lambda\hat{x} + \mathrm{i}\hat{p}) = \mu a + \nu a^{+} \\ B^{+} &= \frac{1}{\sqrt{2\lambda_1}}(\lambda^{*}\hat{x} - \mathrm{i}\hat{p}) = \mu^{*}a^{+} + \nu^{*}a \end{aligned} \tag{13}$$

where a, a^{+} are the annihilation and creation operators and

$$\mu = \frac{1}{2\sqrt{\lambda_1}}(\lambda + 1) \qquad \nu = \frac{1}{2\sqrt{\lambda_1}}(\lambda - 1) \qquad |\mu|^2 - |\nu|^2 = 1 \tag{14}$$

the Yuen [15] operators. The uncertainty principle is given by the relation

$$(\Delta x)(\Delta p) = \frac{1}{2}|\mu+\nu||\mu-\nu| = \frac{1}{2}\sqrt{1 + \frac{\lambda_2^2}{\lambda_1^2}} = \frac{1}{2}\frac{|\lambda|}{\lambda_1} \geqslant \frac{1}{2} \tag{15}$$

that coincides with the formula (12).

Another example with interest is also defined by

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + \frac{z}{2}\sinh x \qquad A^{+} = x \qquad [A, A^{+}] = 1 \qquad \mathrm{Re}\,z > 0 \tag{16}$$

with the eigenfunctions

$$\Psi_n(x) = N_n\, \mathrm{e}^{-\frac{z}{2}\cosh x} x^n \tag{17}$$

which satisfy the equation

$$\left(x\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{z}{2}\sinh x\right) + \frac{1}{2}\right)\Psi_n(x) = \left(n + \frac{1}{2}\right)\Psi_n(x) \tag{18}$$

and the normalization factor N_n is given by the relation

$$N_n^2 \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{z+z^*}{2}\cosh x} x^{2n}\,\mathrm{d}x = 2N_n^2 \int_0^{+\infty} \mathrm{e}^{-z_1\cosh x}\, x^{2n}\,\mathrm{d}x = 1. \tag{19}$$

From the integral representation of the modified Bessel function [16] $K_\nu(z)$, i.e.

$$K_\nu(z_1) = \int_0^\infty \mathrm{e}^{-z_1\cosh x}\cosh \nu\, x\,\mathrm{d}x \tag{20}$$

we obtain

$$\left(\frac{\partial^{2n} K_\nu(z_1)}{\partial \nu^{2n}}\right)_{\nu=0} = \int_0^\infty \mathrm{e}^{-z_1\cosh x} x^{2n}\,\mathrm{d}x \tag{21}$$

and the normalization factor takes the form

$$N_n = \frac{1}{\sqrt{2}}\left[\left(\frac{\partial^{2n}}{\partial\nu^{2n}}K_\nu(z_1)\right)_{\nu=0}\right]^{-\frac{1}{2}}. \tag{22}$$

The mean values are

$$\bar{x} = 0 \qquad \bar{p} = 0 \tag{23}$$

$$\bar{x}^2 = \left[\left(\frac{\partial^{2(n+1)}}{\partial\nu^{2(n+1)}}K_\nu(z_1)\right)_{\nu=0}\right]\left[\left(\frac{\partial^{2n}}{\partial\nu^{2n}}K_\nu(z_1)\right)_{\nu=0}\right]^{-1} \tag{24}$$

$$\begin{aligned}\bar{p}^2 = \frac{1}{2}\left[\left(\frac{\partial^{2n}}{\partial\nu^{2n}}K_\nu(z_1)\right)_{\nu=0}\right]^{-1}&\left\{\frac{z_1}{2}\left(\frac{\partial^{2n}}{\partial\nu^{2n}}[K_{\nu+1}(z_1)+K_{\nu-1}(z_1)]\right)_{\nu=0}\right.\\ &\left.+2n^2\left(\frac{\partial^{2(n-1)}}{\partial\nu^{2(n-1)}}K_\nu(z_1)\right)_{\nu=0} - \frac{z_1^2-z_2^2}{8}\left(\frac{\partial^2}{\partial\nu^{2n}}[K_{\nu+2}(z_1)+K_{\nu-1}(z_1)]\right)_{\nu=0}\right\}\end{aligned} \tag{25}$$

and the uncertainty principle for $n = 0$ has the form

$$(\Delta x)(\Delta p) = \frac{1}{2}\frac{|z|}{z_1}\frac{1}{K_0(z_1)}\left[z_1K_1(z_1)\left(\frac{\partial^2}{\partial\nu^2}K_\nu(z_1)\right)_{\nu=0}\right]^{\frac{1}{2}}. \tag{26}$$

From the above results, we see that many examples exist for non-Hermitian Hamiltonians $H = A^+A + \frac{1}{2}$ with the same real spectrum (and with the normalization condition to be valid).

According to the paper [14], where we have generalized the usual coherent and squeezed states for non-unitary Weyl and squeeze operators, we shall prove with the help of the non-Hermitian harmonic oscillator with two complex parameters, the existence of a discrete complex spectrum or of a real spectrum for specific cases.

3. Non-unitary squeeze operator

We consider the unitary squeeze operator

$$S(z) = \mathrm{e}^{\frac{1}{2}(za^{+2}-z^*a^2)} \tag{27}$$

where the operators a and a^+ are adjoint operators with the commutation relation

$$[a, a^+] = 1. \tag{28}$$

For the case where the operators a, a^+ are non-adjoint, i.e. $(a^+)^\dagger \neq a$ and the relation (28) holds, the squeeze operator (27) is non-unitary. For example, $a = \frac{\mathrm{d}}{\mathrm{d}x}$, $a^+ = x$ the operator (27) is non-unitary

$$S(z) = \mathrm{e}^{\frac{1}{2}(zx^2 - z^* \frac{\mathrm{d}^2}{\mathrm{d}x^2})} \qquad S(z)S^\dagger(z) \neq 1. \tag{29}$$

In the following, we consider the nonHermitian harmonic oscillator with two complex parameters z, b with $\mathrm{Re}\, z > 0$, $\mathrm{Re}\, b > 0$, i.e.

$$H = \tfrac{1}{2}(z\hat{x}^2 + b\hat{p}^2) \tag{30}$$

$$H^\dagger = \tfrac{1}{2}(z^*\hat{x}^2 + b^*\hat{p}^2) \tag{31}$$

with $H \neq H^\dagger$.

The above operators in x- and p-representation take the form

$$H = \frac{1}{2}\left(zx^2 - b\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \qquad H^\dagger = \frac{1}{2}\left(z^*x^2 - b^*\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \tag{32}$$

$$H = \frac{1}{2}\left(-z\frac{\mathrm{d}^2}{\mathrm{d}p^2} + bp^2\right) \qquad H^\dagger = \frac{1}{2}\left(-z^*\frac{\mathrm{d}^2}{\mathrm{d}p^2} + b^*p^2\right). \tag{33}$$

According to the above operators, we can define the corresponding non-unitary squeeze operators with the two complex parameters z and b as

$$S_x(z, b) = \mathrm{e}^{\frac{1}{2}(zx^2 - b\frac{\mathrm{d}^2}{\mathrm{d}x^2})} \qquad {S_x}^\dagger(z, b) = \mathrm{e}^{\frac{1}{2}(z^*x^2 - b^*\frac{\mathrm{d}^2}{\mathrm{d}x^2})} \tag{34}$$

$$S_p(z, b) = \mathrm{e}^{\frac{1}{2}(bp^2 - z\frac{\mathrm{d}^2}{\mathrm{d}p^2})} \qquad {S_p}^\dagger(z, b) = \mathrm{e}^{\frac{1}{2}(b^*p^2 - z^*\frac{\mathrm{d}^2}{\mathrm{d}p^2})}. \tag{35}$$

Since the operators $S_x(z, b)$ and $\frac{1}{2}\left(zx^2 - b\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)$ commute, the eigenfunctions of $S_x(z, b)$ coincide with the corresponding eigenfunctions of the operator $\frac{1}{2}\left(zx^2 - b\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)$. Therefore, we have to solve the equation

$$\left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{E}{b} - \frac{1}{2}\frac{z}{b}x^2\right)\Psi_n(x) = 0. \tag{36}$$

The solution of the above equation has the form

$$\Psi_n(x) = N_n\,\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{z}{b}}x^2} H_n\left(\sqrt[4]{\frac{z}{b}}x\right) \tag{37}$$

with the eigenvalues

$$E = \sqrt{zb}\left(n + \tfrac{1}{2}\right). \tag{38}$$

The eigenfunctions

$${\Psi_n}^*(x) = N_n\,\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{z^*}{b^*}}x^2} H_n\left(\sqrt[4]{\frac{z^*}{b^*}}x\right) \tag{39}$$

lead to the eigenvalues

$$E = \sqrt{z^*b^*}\left(n + \tfrac{1}{2}\right) \tag{40}$$

corresponding to the operator $H^\dagger$. The polynomials $H_n(u)$ are the Hermite polynomials and the normalization factor N_n is given by

$$N_n = \sqrt[4]{\frac{\sqrt{zb^*} + \sqrt{z^*b}}{2\pi|b|}}\frac{1}{\sqrt{2^n n!}}\left(P_n\left(\frac{2\sqrt{|z||b|}}{\sqrt{zb^* + bz^* + 2|z||b|}}\right)\right)^{-\frac{1}{2}}. \tag{41}$$

For the calculation of the resulting integrals, we use the formula [17]

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-sx^2} H_m(\beta_1 x) H_n(\beta_2 x)\,\mathrm{d}x = n!\left(\frac{\pi}{s}\right)^{\frac{1}{2}}\left(-\frac{1-\gamma_1}{1-\gamma_2}\right)^{\frac{m-n}{4}} \times \left(2(\gamma_1+\gamma_2-1)^{\frac{1}{2}}\right)^{\frac{m+n}{2}} P_{\frac{m+n}{2}}^{\frac{|m-n|}{2}}\left[\left(\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-1}\right)^{\frac{1}{2}}\right] \tag{42}$$

where $s > 0$, $\gamma_1 = \frac{\beta_1^2}{s}$, $\gamma_2 = \frac{\beta_2^2}{s}$, $m \geqslant n$, $P_n^l(u)$ are the associated Legendre polynomials [18] and $P_n(u)$ the usual Legendre polynomials. The numbers m and n in the above formula are of the same parity.

The eigenfunctions in the p-representation take the form

$$\varphi_n(p) = \tilde{N}_n\,\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{b}{z}}p^2} H_n\left(\sqrt[4]{\frac{b}{z}}\,p\right) \tag{43}$$

where

$$\tilde{N}_n = \sqrt[4]{\frac{\sqrt{zb^*}+\sqrt{z^*b}}{2\pi|z|}}\frac{1}{\sqrt{2^n n!}}\left(P_n\left(\frac{2\sqrt{|b||z|}}{\sqrt{zb^*+bz^*+2|z||b|}}\right)\right)^{-\frac{1}{2}} \tag{44}$$

with the same eigenvalues (38). For the case $b = z^*$, we obtain the real spectrum

$$E = |z|\left(n+\tfrac{1}{2}\right) \tag{45}$$

with the eigenfunctions

$$\Psi_n(x) = \sqrt[4]{\frac{z_1}{\pi|z|}}\frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{P_n\left(\frac{|z|}{z_1}\right)}}\,\mathrm{e}^{-\frac{z}{2|z|}x^2} H_n\left(\sqrt{\frac{z}{|z|}}\,x\right) \tag{46}$$

$$\varphi_n(p) = \sqrt[4]{\frac{z_1}{\pi|z|}}\frac{1}{\sqrt{2^n n!}}\frac{1}{\sqrt{P_n\left(\frac{|z|}{z_1}\right)}}\,\mathrm{e}^{-\frac{z^*}{2|z|}p^2} H_n\left(\sqrt{\frac{z^*}{|z|}}\,p\right). \tag{47}$$

The main characteristic point, in the case of the non-Hermitian harmonic oscillator in x- and p-representation, is that the eigenfunctions are not orthonormal, namely

$$\int_{-\infty}^{+\infty} \Psi_n^*(x)\Psi_l(x)\,\mathrm{d}x = \sqrt{\frac{n!}{l!}}\frac{P_{(l+n)/2}^{|l-n|/2}\left(\frac{|z|}{z_1}\right)}{\sqrt{P_n\left(\frac{|z|}{z_1}\right)P_l\left(\frac{|z|}{z_1}\right)}} \neq \delta_{\mathrm{ln}}. \tag{48}$$

Results of the same form are obtained for the eigenfunctions of momentum space.

By using the eigenfunctions (46) in x- and p-representation, we can find the mean values $\bar{x}$, $\bar{p}$, $\bar{x}^2$, $\bar{p}^2$ and the dispersion relations $(\Delta x)^2$, $(\Delta p)^2$. After some algebra, we obtain $\bar{x} = 0$, $\bar{p} = 0$,

$$\bar{x}^2 = \bar{p}^2 = \frac{1}{2}\frac{(n+1)P_{n+1}\left(\frac{|z|}{z_1}\right) + nP_{n-1}\left(\frac{|z|}{z_1}\right) + P_n^1\left(\frac{|z|}{z_1}\right)}{P_n\left(\frac{|z|}{z_1}\right)} \tag{49}$$

and

$$(\Delta x)(\Delta p) = \frac{1}{2}\frac{(n+1)P_{n+1}\left(\frac{|z|}{z_1}\right) + nP_{n-1}\left(\frac{|z|}{z_1}\right) + P_n^1\left(\frac{|z|}{z_1}\right)}{P_n\left(\frac{|z|}{z_1}\right)}. \tag{50}$$

For $n = 0$, the above relations yield

$$(\Delta x)(\Delta p) = \frac{1}{2}\sqrt{1+\frac{z_2^2}{z_1^2}} \geqslant \frac{1}{2}. \tag{51}$$

From the operators (32), we can construct the adjoint Yuen [15] operators Γ and Γ^+ and obtain the same uncertainty principle, i.e.

$$\begin{aligned}\Gamma &= (\sqrt{zb^*}+\sqrt{z^*b})^{-\frac{1}{2}}(\sqrt{z}\hat{x}+\mathrm{i}\sqrt{b}\hat{p}) = \mu a+\nu a^+\\ \Gamma^+ &= (\sqrt{zb^*}+\sqrt{z^*b})^{-\frac{1}{2}}(\sqrt{z^*}\hat{x}-\mathrm{i}\sqrt{b^*}\hat{p}) = \mu^* a^+ +\nu^* a\end{aligned} \tag{52}$$

with

$$\begin{aligned}&[\Gamma,\Gamma^+]=1 \qquad \mu=[2(\sqrt{zb^*}+\sqrt{z^*b})]^{-\frac{1}{2}}(\sqrt{z}+\sqrt{b})\\ &\nu=[2(\sqrt{zb^*}+\sqrt{z^*b})]^{-\frac{1}{2}}(\sqrt{z}-\sqrt{b}) \qquad |\mu|^2-|\nu|^2=1.\end{aligned} \tag{53}$$

The uncertainty principle is given by the relation

$$(\Delta x)(\Delta p)=\frac{1}{2}|\mu+\nu||\mu-\nu|=\frac{1}{2}\sqrt{\frac{4|z||b|}{zb^*+bz^*+2|z||b|}}=\frac{1}{2}\sqrt{\frac{2|z||b|}{z_1b_1+z_2b_2+|z||b|}}. \tag{54}$$

For $b^*=z$, we obtain the relation (51). The above formula results easily from the eigenfunctions (37), (43) for $n=0$, so that

$$\overline{\hat{x}^2}=(\Delta x)^2=\frac{|b|}{\sqrt{z^*b}+\sqrt{zb^*}} \qquad \overline{\hat{p}^2}=(\Delta p)^2=\frac{|z|}{\sqrt{z^*b}+\sqrt{zb^*}} \tag{55}$$

and we obtain the relation (54).

From the above results, we conclude that where we have a discrete complex spectrum (38) for non-unitary squeeze operators (34) and for $b^*=z$, the eigenvalues are real (45), i.e. $|z|\left(n+\frac{1}{2}\right)$ and there exist eigenstates for these non-unitary squeeze operators. Instead of the relation $b^*=z$, there also exists the relation

$$z_1b_2+z_2b_1=0 \qquad \text{or} \qquad b_2=-\frac{z_2b_1}{z_1} \tag{56}$$

where the eigenvalues of (38) are real and positive, i.e.

$$E=\sqrt{z_1b_1-z_2b_2}\left(n+\frac{1}{2}\right)=\sqrt{\frac{b_1}{z_1}}|z|\left(n+\frac{1}{2}\right) \tag{57}$$

which, for $b_1=z_1$, coincides with the relation (45).

The discrete complex spectrum for the non-unitary squeeze operator has the form

$$S_x(z,b)\Psi_n(x)=\mathrm{e}^{\frac{1}{2}\sqrt{zb}(n+\frac{1}{2})}\psi_n(x) \tag{58}$$

where $\psi_n(x)$ are the eigenfunctions (37). For $z=|z|\,\mathrm{e}^{\mathrm{i}\theta}$, $b=|b|\,\mathrm{e}^{\mathrm{i}\varphi}$, the above equation yields

$$S_x(z,b)\Psi_n(x)=\mathrm{e}^{\frac{1}{2}\sqrt{|z||b|}(\cos\frac{\theta+\varphi}{2}+\mathrm{i}\sin\frac{\theta+\varphi}{2})}\psi_n(x) \tag{59}$$

and the exponent represents a complex phase.

Because of the relation (56), for $\theta+\varphi=0$, we obtain the real phase $\mathrm{e}^{\frac{1}{2}\sqrt{\frac{b_1}{z_1}}|z|(n+\frac{1}{2})}$ and for $\theta+\varphi=\pi$, the imaginary phase $\mathrm{e}^{\mathrm{i}\frac{1}{2}\sqrt{\frac{b_1}{z_1}}|z|(n+\frac{1}{2})}$.

Finally, we consider a simple example, which is connected to the *PT* symmetry by the non-Hermitian Hamiltonian

$$\mathcal{H}=\frac{p^2}{2m}+\frac{m\omega^2}{2}(x^2+2\lambda x) \tag{60}$$

with $\mathrm{Re}(m)>0$ and λ complex.

For the case m and λ real, the above operator is Hermitian with the eigenfunctions

$$\Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \mathrm{e}^{-\frac{m\omega}{2\hbar}(x+\lambda)^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x+\lambda)\right) \tag{61}$$

and the eigenvalues

$$E_n(\lambda) = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{m\omega^2\lambda^2}{2}. \tag{62}$$

For the conservation of PT symmetry, the following relation holds

$$P : x \to -x-2\lambda \quad p \to -p \qquad T : p \to -p \quad \mathrm{i} \to -\mathrm{i}. \tag{63}$$

For the case of complex m and λ, we obtain the condition

$$\mathcal{H}^* = \mathcal{H}^\dagger = \frac{p^2}{2m^*} + \frac{m^*\omega^2}{2}(x^2 + 2\lambda^* x) \tag{64}$$

where $\mathcal{H}^*$ is the conjugate of $\mathcal{H}$ and $\mathcal{H}^\dagger$ is the adjoint.

The Hamiltonian (60) in x-representation satisfies the wave equation

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Psi_n}{\mathrm{d}x^2} + \left(E + \frac{m\omega^2\lambda^2}{2} - \frac{m\omega^2}{2}(x+\lambda)^2\right)\Psi_n(x) = 0 \tag{65}$$

with the solution

$$\Psi_n(x) = N_n\,\mathrm{e}^{-\frac{m\omega}{2\hbar}(x+\lambda)^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x+\lambda)\right) \tag{66}$$

and the complex eigenvalues are

$$E_n(\lambda) = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{m\omega^2\lambda^2}{2}. \tag{67}$$

The necessary condition so that the above eigenvalues are real, is

$$\mathrm{Im}(m\lambda^2) = m_2\left(\lambda_1^2 - \lambda_2^2\right) + 2m_1\lambda_1\lambda_2 = 0 \qquad \text{or} \qquad m_2 = -\frac{2m_1\lambda_1\lambda_2}{\lambda_1^2-\lambda_2^2} \tag{68}$$

and we obtain the real eigenvalues

$$E_n(\lambda_1,\lambda_2) = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{m_1\omega^2}{2}\frac{\left(\lambda_1^2+\lambda_2^2\right)^2}{\lambda_1^2-\lambda_2^2}. \tag{69}$$

The normalization factor N_n for the real eigenvalues (69) after some algebra takes the form

$$N_n = \left(\frac{m_1\omega}{\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{m_1\omega}{2\hbar}\frac{\lambda_2^2}{\lambda_1^2}\delta^2} \left\{\sum_{\ell=0}^{n}\binom{n}{\ell}^2 H_\ell(K)H_\ell(K^*)\left(\frac{4|m|}{m_1}\right)^{n-\ell}\Gamma\left(n-\ell+\frac{1}{2}\right)\right\}^{-\frac{1}{2}} \tag{70}$$

where

$$K = \sqrt{\frac{m\omega}{\hbar}}(\lambda-\delta) \qquad \delta = \lambda_1\frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2-\lambda_2^2}. \tag{71}$$

From the above results, we see that the spectrum (69) is real, even though the Hamiltonian (60) in this case is not PT symmetric.

By using the eigenfunctions for the ground state $n = 0$ in x- and p-representation, we obtain the dispersion relations

$$(\Delta x)^2 = \frac{\hbar}{2m_1\omega} \qquad (\Delta p)^2 = \frac{\hbar|m|^2\omega}{2m_1} \quad \text{and} \quad (\Delta x)(\Delta p) \geqslant \frac{\hbar}{2}\frac{|m|}{m_1}. \tag{72}$$

The case of real mass and complex λ is particularly interesting, with the complex eigenvalues

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{m\omega^2}{2}(\lambda_1 + \mathrm{i}\lambda_2)^2 \tag{73}$$

and the following eigenfunctions in x- and p-representation

$$\Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{m\omega}{2\hbar}\lambda_2^2}\left[2^n n! L_n\left(-2\frac{m\omega}{2\hbar}\lambda_2^2\right)\right]^{-\frac{1}{2}} \mathrm{e}^{-\frac{m\omega}{2\hbar}(x+\lambda)^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}(x+\lambda)\right) \tag{74}$$

$$\Phi_n(x) = \left(\frac{1}{\pi\hbar m\omega}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{m\omega}{2\hbar}\lambda_2^2}\left[2^n n! L_n\left(-2\frac{m\omega}{2\hbar}\lambda_2^2\right)\right]^{-\frac{1}{2}} \mathrm{e}^{-\frac{p^2}{2m\hbar\omega}+\mathrm{i}\frac{\lambda p}{\hbar}} H_n\left(\sqrt{\frac{1}{\hbar m\omega}}p\right) \tag{75}$$

where $L_n\left(-2\frac{m\omega}{2\hbar}\lambda_2^2\right)$ are the Laguerre polynomials.

4. Conclusion

In conclusion, we have enlarged the method of [1] for the non-Hermitian harmonic oscillator with complex parameters, where the operators A and A^+ are not adjoint $(A^+)^\dagger \neq A$ and satisfy the relation $[A, A^+] = 1$. From the results, we see that many examples exist for the non-Hermitian Hamiltonian $\mathcal{H} = A^+A + \frac{1}{2}$ with the same real spectrum. For the non-unitary squeeze operators (34), and for $a = \frac{\mathrm{d}}{\mathrm{d}x}$, $a^+ = x$, $[a, a^+] = 1$, we obtain the discrete complex eigenvalues (38). According to the relation (56), we get a discrete real spectrum (57), even though the Hamilton operator (32) is not PT symmetric. Also the spectrum (69) is real even though the Hamiltonian (60) is not PT symmetric. For the case of real mass and complex eigenvalues (73), the corresponding operator is not PT symmetric.

For the non-unitary squeeze operator $S(z, w) = \exp\left[\frac{1}{2}\left(za^{+^2} - w^*a^2\right)\right]$ and the non-unitary Weyl operator $D(\alpha, \beta) = \exp(\alpha a^+ - \beta^* a)$ in this case a, a^+ are adjoint, we have extended the usual definition of the squeezed states by the form [14] $|z, w, \alpha, \beta\rangle = D(\alpha, \beta)S(z, w)|0\rangle$, from which we obtain

$$|z, w, \alpha, \beta\rangle = \sqrt{\frac{\cosh|z|}{\cosh\sqrt{zw^*}}}\,\mathrm{e}^{\frac{1}{2}\alpha(\alpha^*-\beta^*)} \exp\left\{\frac{z}{2}\left[\frac{\tanh\sqrt{zw^*}}{\sqrt{zw^*}}(a^+ - \beta^*)^2 - \frac{\tanh|z|}{|z|}(a^+ - \alpha^*)^2\right]\right\}|z, \alpha\rangle. \tag{76}$$

For $\beta = \alpha$, $w = z$, the above squeeze states coincide with the usual $|z, \alpha\rangle$.

Also, from the non-Hermitian operator (34), we can construct the Yuen operators (52) (two photon states) and can easily obtain the uncertainty relation. More details and applications relative to squeeze states are referred to in the recent review article by Dodonov [19].

Finally, we remark that the problem of the existence of eigenfunctions and real discrete eigenvalues for the non-unitary squeeze operators is of physical interest. We hope that the present ideas will find many applications.

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